Integrable and non-integrable equations with peakons

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Abstract

We consider a one-parameter family of non-evolutionary partial differential equations which includes the integrable Camassa-Holm equation and a new integrable equation first isolated by Degasperis and Procesi. A Lagrangian and Hamiltonian formulation is presented for the whole family of equations, and we discuss how this fits into a bi-Hamiltonian framework in the integrable cases. The Hamiltonian dynamics of peakons and some other special finite-dimensional reductions are also described.

1 Introduction

In this note we consider the following one-parameter family of partial differential equations (PDEs):

$$u_t - u_{xxt} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx} \tag{1.1}$$

(the parameter b is constant). In the particular case $b = 2$ this becomes the dispersionless version of the integrable Camassa-Holm equation, which is

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}. \tag{1.2}$$

With the inclusion of linear dispersion terms the equation (1.2) takes the form

$$u_t + c_0 u_x + \gamma u_{xxx} - \alpha^2 u_{xxt} = 2u_x u_{xx} + uu_{xxx} \tag{1.3}$$

(c_0, γ and α are constant parameters), and in this form it was derived as an approximation to the incompressible Euler equations, and found to be completely integrable with a Lax pair and associated bi-Hamiltonian structure [1, 2].

Degasperis and Procesi [3] applied the method of asymptotic integrability to test a many-parameter family of equations generalizing (1.3), and found that only three equations passed the test up to third order in the asymptotic expansion, namely KdV, Camassa-Holm and one new equation. After rescaling and applying a Galilean transformation, the new equation may be written in dispersionless form as

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}. \tag{1.4}$$

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Henceforth we refer to (1.4) as the Degasperis-Procesi equation, and observe that it is the $b = 3$ case of the equation (1.1). In a recent article [4] we proved the integrability of the Degasperis-Procesi equation by providing a Lax pair, and derived two infinite sequences of conservation laws. The key to our construction was to use a reciprocal transformation connecting the equation (1.4) with a negative flow in the Kaup-Kupershmidt hierarchy.

Both the Camassa-Holm equation (1.2) and the Degasperis-Procesi equation (1.4) display the weak Painlevé property [12], with algebraic branching in their solutions. Thus when Gilson and Pickering applied the standard Painlevé tests to a class of equations including the family (1.1), these two integrable equations were excluded because of the branching, and they found that no equations passed the tests. In a forthcoming work [5] we show that after applying a reciprocal transformation to the family (1.1) it is possible to use Painlevé analysis effectively, and the requirement of only pole singularities immediately isolates the two integrable cases $b = 2, 3$. Because the equations (1.1) are non-evolutionary, the standard symmetry approach of Shabat et al [9] does not apply. However, recently Mikhailov and Novikov developed a powerful extension of the symmetry classification method [10], and applying this to the equations (1.1) they found that only the cases $b = 2, 3$ could possess infinitely many commuting symmetries, and so only these two cases are integrable.

One of the novel properties of the Camassa-Holm equation is that it admits exact solutions in terms of a superposition of an arbitrary number of peakons, or peaked solitons. The N-peakon solution is just

$$u(x,t) = \sum_{j=1}^{N} p_j(t) e^{-|x-q_j(t)|}, \tag{1.5}$$

where the positions $q_j(t)$ of the peaks and their momenta $p_j(t)$ satisfy an associated dynamical system taking the canonical Hamiltonian form

$$\dot{p}_j = -\frac{\partial h}{\partial q_j}, \qquad \dot{q}_j = \frac{\partial h}{\partial p_j}, \tag{1.6}$$

with

$$h = \frac{1}{2} \sum_{j,k=1}^{N} p_j p_k \, e^{-|q_j - q_k|}. \tag{1.7}$$

It is shown in [1, 2] that this is a completely integrable finite-dimensional system, with an $N \times N$ Lax pair. The integrability of the Camassa-Holm peakon dynamics further received an r-matrix interpretation in [11].

One of our observations in [4] was that all of the equations in the family (1.1) admit N-peakon solutions in the form (1.5), but the dynamical system for q_j, p_j only takes the canonical Hamiltonian form (1.6) in the particular case $b = 2$. In the following we construct a Lagrangian (for $b \neq 0$) and associated Hamiltonian structure, depending on the parameter b, for each equation in the family (1.1). We then reduce this onto the finite-dimensional submanifold of peakon solutions for each N to obtain a new non-canonical Poisson structure and associated Hamiltonian form for the peakon dynamics. It turns out the two-body dynamics ($N = 2$) is Liouville integrable for any value of b, but for arbitrary N we expect integrability only for $b = 2, 3$ since these are the reductions of integrable PDEs. An $N \times N$ matrix Lax matrix pair for the $b = 3$ peakon dynamics was presented in [4], but at that stage we were still ignorant of the correct Poisson structure.

2 Lagrangian form and Hamiltonian operator

In the following we consider solutions of (1.1) on the whole x-axis vanishing at infinity, and all integrals will be assumed to be between $\pm\infty$, although (with suitable modification) most of the results hold for solutions with constant background or on a periodic domain. Some of the operations below only make sense for sufficiently smooth real functions u, but later we will want to apply some formulae to weak solutions such as the peakons, in which case we will find that certain manipulations are valid also for distributions.

It is convenient to introduce the quantity

$$m = u - u_{xx},$$

which is just the Helmholtz operator acting on u, so that the family of equations (1.1) may be rewritten in the form

$$m_t + um_x + b\,u_x m = 0. \tag{2.8}$$

For any value of b this is a conservation law, i.e.

$$m_t + \left(\frac{(b-1)}{2}(u^2 - u_x^2) + um\right)_x = 0,$$

while for all $b \neq 0$ there are at least three conserved quantities, namely

$$\int m\,dx, \quad \int m^{1/b}\,dx, \quad \int m^{-1/b}\left(\frac{m_x^2}{b^2m^2} + 1\right)dx. \tag{2.9}$$

For the integrable cases $b = 2$ [1, 2, 13] and $b = 3$ [4] there are infinitely many conserved quantities.

For the second quantity appearing in (2.9) above, the associated conservation law is conveniently written as

$$p_t + (pu)_x = 0, \qquad m = -p^b. \tag{2.10}$$

With this further rewriting of the equation (2.8), or equivalently (1.1), for $b \neq 0$ we introduce a potential η such that

$$\eta_x = p, \qquad \eta_t = -pu.$$

Then in terms of this potential we find that (apart from the particular cases $b = 0, 1$) the equation (1.1) may be derived from the following action:

$$S \equiv \int\int \mathcal{L}\,dx\,dt = \int\int \left(\frac{1}{2}\frac{\eta_t}{\eta_x}\left[(\log\eta_x)_{xx} + 1\right] + \frac{\eta_x^b}{b-1}\right)dx\,dt. \tag{2.11}$$

When $b = 1$ a similar Lagrangian formulation is valid, but the Lagrange density must be modified to

$$\mathcal{L} = \frac{1}{2}\eta_t\eta_x^{-1}\left[(\log\eta_x)_{xx} + 1\right] + \eta_x\log\eta_x.$$

Starting from the Lagrangian we can apply a Legendre transformation in the usual way. The conjugate momentum is

$$\zeta \equiv \frac{\partial\mathcal{L}}{\partial\eta_t} = \frac{1}{2\eta_x}\left[(\log\eta_x)_{xx} + 1\right],$$

and (for $b \neq 1$) the Hamiltonian is

$$H = \int(\zeta\eta_t - \mathcal{L})\,dx = -\frac{1}{b-1}\int\eta_x^b\,dx = \frac{1}{b-1}\int m\,dx. \tag{2.12}$$

Having applied the Legendre transformation we then find that for any b the PDE (2.8) can be written in Hamiltonian form as

$$m_t = \hat{B}\frac{\delta H}{\delta m}, \tag{2.13}$$

where the operator

$$\hat{B} = -(bm\partial_x + m_x)(\partial_x - \partial_x^3)^{-1}(bm\partial_x + (b-1)m_x) \tag{2.14}$$

is skew-symmetric and satisfies the Jacobi identity (for a proof see [5]). In the case $b = 1$ the Legendre transformation gives a different result: $\int m\,dx$ is a Casimir for the operator $\hat{B}$, and instead the Hamiltonian may be taken as

$$H = \int m \log m \, dx.$$

The formula (2.13) is even valid in the case $b = 0$, when the Lagrangian formulation breaks down.

For the integrable case $b = 2$, the Camassa-Holm equation (1.2), there are two local Hamiltonian structures [1, 2], given by

$$B_0 = -\partial_x(1 - \partial_x^2), \qquad B_1 = -(m\partial_x + \partial_x m),$$

and the compatible operator (2.14) is the first nonlocal Hamiltonian structure obtained by applying the recursion operator $R = B_1 B_0^{-1}$ to B_1, i.e.

$$B_2 \equiv B_1 B_0^{-1} B_1 = \hat{B}|_{b=2}.$$

In the other integrable case $b = 3$, namely the Degasperis-Procesi equation (1.4), there is only one local Hamiltonian structure, and the operator (2.14) gives the second Hamiltonian structure, viz

$$B_0 = -\partial_x(1 - \partial_x^2)(4 - \partial_x^2), \qquad B_1 = \hat{B}|_{b=3}. \tag{2.15}$$

This compatible bi-Hamiltonian pair first appeared in [4]. The nonlocal part of (2.14) may be defined precisely via

$$(\partial_x - \partial_x^3)^{-1} f(x) = \frac{1}{2}\int_{-\infty}^{\infty} G(x-y) f(y)\, dy, \qquad G(x) = sgn(x)(1 - e^{-|x|}). \tag{2.16}$$

The Jacobi identity for (2.14) reduces to a single functional equation satisfied by the kernel G [5].

3 Peakon dynamics and special solutions

The peakon solutions of (1.1) take the form (1.5) for any b, in which case the dependent variable m is a sum of Dirac delta functions,

$$m(x,t) = 2\sum_{j=1}^{N} p_j(t)\delta(x - q_j(t)). \tag{3.17}$$

Introducing the even kernel (Green's function of the Helmholtz operator)

$$g(x) = e^{-|x|}, \qquad G(x) = \int_0^x g(s)\, ds,$$

the dynamical system for the peakons may be conveniently written as

$$\begin{aligned} \frac{dq_j}{dt} &= \sum_{k=1}^N p_k g(q_j - q_k), \\ \frac{dp_j}{dt} &= -(b-1)\sum_{k=1}^N p_j p_k g'(q_j - q_k). \end{aligned} \tag{3.18}$$

Clearly this takes the canonical Hamiltonian form (1.6) only for $b = 2$.

In order to obtain the correct Poisson structure on the reduced peakon phase space, for any $b \neq 1$ we can use the Poisson bracket defined by (2.14), which gives (up to rescaling)

$$\{m(x), m(y)\} = \frac{1}{(b-1)}\Big(G(x-y)m_x(x)m_x(y) + bG'(x-y)m(x)m_x(y)$$
$$- bG'(x-y)m(y)m_x(x) - b^2G''(x-y)m(x)m(y)\Big). \tag{3.19}$$

Substituting the peakon expression (3.17) into (3.19), we are able to calculate the Poisson bracket on the reduced ($2N$-dimensional) phase space as

$$\{p_j, p_k\} = -(b-1)G''(q_j - q_k)p_j p_k, \quad \{q_j, p_k\} = p_k G'(q_j - q_k),$$
$$\{q_j, q_k\} = 1/(b-1)G(q_j - q_k). \tag{3.20}$$

The Jacobi identity for this non-canonical Poisson bracket follows from the same functional equation for G that arises for the operator (2.14) (see [5]). The Hamiltonian for the dynamical system (3.18) is found simply by substituting (3.17) into the PDE Hamiltonian (2.12), so that after scaling we have

$$\tilde{h} = (b-1)H/2 = \sum_{j=1}^N p_j.$$

Having obtained the reduction of the Poisson structure for the PDE, we see that the peakon dynamical system (3.18) takes the Hamiltonian form

$$\frac{dq_j}{dt} = \{q_j, \tilde{h}\}, \qquad \frac{dp_j}{dt} = \{p_j, \tilde{h}\}.$$

Now for any b we also introduce the second quantity

$$k = \frac{1}{2}\sum_{j,k=1}^N p_j p_k \Big(1 - (1 - g(q_j - q_k))^{b-1}\Big).$$

For $b = 2$ this just coincides with the canonical Hamiltonian (1.7) for N Camassa-Holm peakons, $k = h$, and is clearly conserved. For the Degasperis-Procesi case $b = 3$, with the bracket (3.20) k Poisson commutes with $\tilde{h}$ for any N; it is the second conserved quantity obtained from $\mathtt{trace}\,\mathsf{L}^2$, where L is the Lax matrix presented in [4]. However, for any other value of b it seems that k is conserved only in the two-body case $N = 2$. Thus we see that for generic values of the parameter b only the two peakon dynamics is Liouville integrable. The two-body dynamics and associated peakon phase shifts are described in more detail in [5].

The integrability of the two-body dynamics seems to induce remarkable stability properties for the peakons even in the non-integrable regime. Numerical studies of the PDE family (1.1) on a periodic domain [8] show that starting from a Gaussian initial profile for u, for any $b > 1$ a train of peakons emerge and preserve their shape and speed after interaction, undergoing only a phase shift. The peakons are unstable for $b < 1$. For the range $-1 < b < 1$ the

numerical results indicate that "ramps-and-cliffs" profiles, like those appearing as solutions to the Burgers equation, seem to be stable. We do not have an exact analytic expression for such profiles. However, the "ramp" part behaves like the exact similarity solution

$$u(x,t) = \frac{x}{(b+1)t},$$

while for the special case $b = 0$ we find an exact solution in terms of a superposition of N "cliffs,"

$$u(x,t) = c + \sum_{j=1}^{N} a_j G(x - ct - Q_j(t)),$$

with c and the a_j being constants. The Q_j satisfy the Hamiltonian dynamical system

$$\frac{dQ_j}{dt} = \sum_{k \neq j} a_k G(Q_j - Q_k) \equiv \{Q_j, \hat{h}\}, \qquad \hat{h} = \sum_{j=1}^{N} a_j Q_j,$$

where (after scaling) the Poisson bracket reduced from (3.19) is

$$\{Q_j, Q_k\} = G(Q_j - Q_k).$$

The "cliff" dynamics is integrable for $N = 2$ and $N = 3$.

In the remaining range $b < -1$, the numerical results of [8] show that Gaussian initial data splits into a superposition of pulses which are asymptotically stationary. For that parameter range there is a unique vanishing stationary solution given by the exact sech-shaped profile

$$u(x) = \Big(\cosh[(b+1)(x - x_0)/2]\Big)^{2/(b+1)}$$

centred around the arbitrary point x_0. In future work we intend to address the stability properties of the peakons and other exact solutions.

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